Numerical time integration methods for nonsmooth systems.
Part I. Low relative degree and electrical networks applications

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Joint work with Olivier Bonnefon & Bernard Brogliato

The RLC circuit with a diode

Example

A LC oscillator supplying a load resistor through a half-wave rectifier (see figure 1).

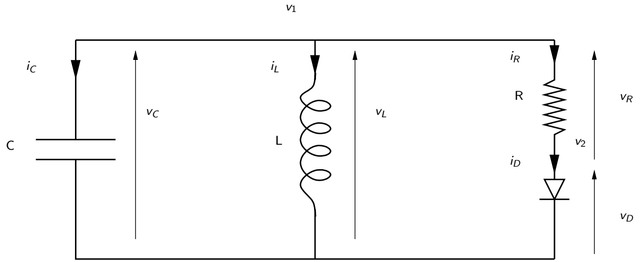
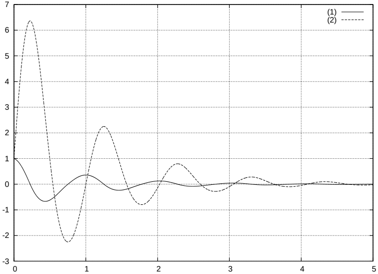


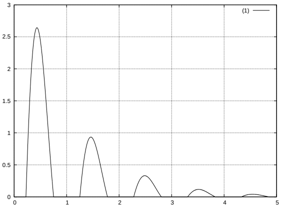
Figure: Electrical oscillator with half-wave rectifier

The RLC circuit with a diode

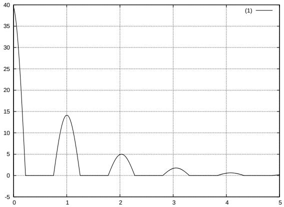
Example



(a) state versus time v_L and i_L



(b) Diode current i_D



(c) Diode voltage v_D

The RLC circuit with a diode

Example

- Kirchhoff laws :

$$\begin{aligned} v_L &= v_C \\ v_R + v_D &= v_C \\ i_C + i_L + i_R &= 0 \\ i_R &= i_D \end{aligned}$$

- Branch constitutive equations for linear devices are :

$$\begin{aligned} i_C &= C\dot{v}_C \\ v_L &= L\dot{i}_L \\ v_R &= Ri_R \end{aligned}$$

- "branch constitutive equation" of the diode

$$0 \in \mathcal{F}(i_D, V_D)$$

The RLC circuit with a diode

Example

The following dynamical system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

$$v_D = v_L - Ri_D$$

$$0 \in \mathcal{F}(v_D, i_D)$$

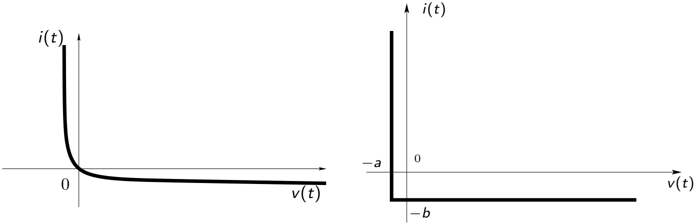
with the state variable $x \triangleq \begin{pmatrix} v_L \\ i_L \end{pmatrix}$ and $\lambda \triangleq i_D$, $y \triangleq v_D$, we get

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \in \mathcal{F}(y, \lambda) \end{cases} \tag{1}$$

Diode behavior

A modeling choice

smooth modeling | nonsmooth modeling



(a) $i(t) = i_s \exp(-\frac{v(t)}{\alpha} - 1)$

(b) $0 \leqslant i(t) + b \perp v(t) + a \geqslant 0$

Figure: Two models of diodes.

Diode behavior

Why a nonsmooth modeling ?

- To avoid stiff nonlinear models by using ideal constraints.
- To model the ideal behavior of switched components without artificial regularization

The diode–bridge rectifier

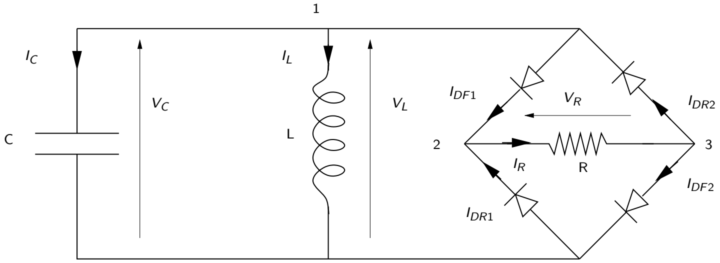


Figure: The Diode-bridge rectifier

The diode–bridge rectifier

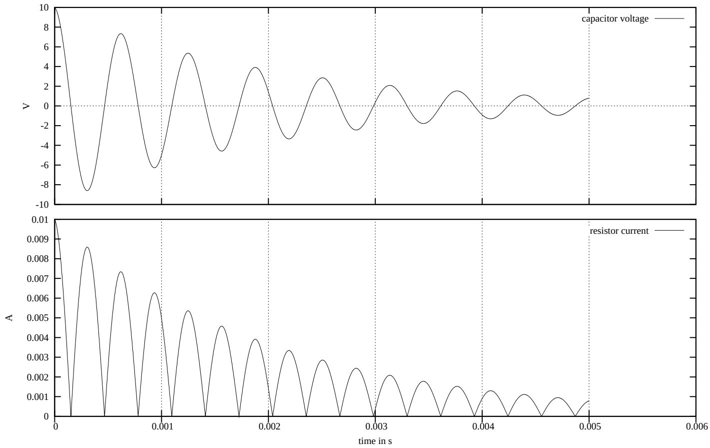


Figure: The Diode-bridge rectifier. Standard results

The diode–bridge rectifier

Differential systems

The dynamical equations are formulated as

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{2}$$

choosing :

$$x = \begin{bmatrix} V_L \\ I_L \end{bmatrix}, \quad \text{and } y = \begin{bmatrix} I_{DR1} \\ I_{DF2} \\ V_2 - V_1 \\ V_1 - V_3 \end{bmatrix}, \quad \lambda = \begin{bmatrix} V_2 \\ -V_3 \\ I_{DF1} \\ I_{DR2} \end{bmatrix}, \tag{3}$$

and with

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1/C & 1/C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

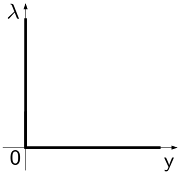
$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{4}$$

A typical example of nonsmooth systems

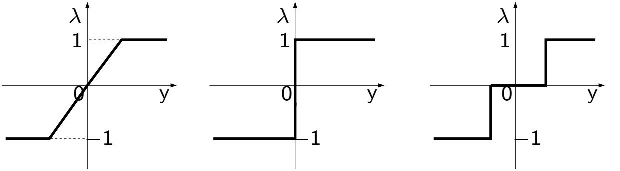
Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{5}$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$
$C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$, for m constraints.



Piecewise linear systems



Saturation   Relay   Relay with dead zone

A slightly more general class of nonsmooth systems

Differential inclusion into normal cones

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ -y \in N_K(\lambda) \end{cases} \tag{6}$$

where K is a convex set and $N_K(\lambda)$ stands for the normal cone to K taken at λ

Usual examples for K

- $K = \mathbb{R}^m$, then we obtain linear time invariant DAE

$$-y \in N_{\mathbb{R}^m}(\lambda) \Longleftrightarrow y = 0, \quad \lambda \in \mathbb{R}^m \tag{7}$$

- $K = \mathbb{R}^m_+$, then we obtain Linear Complementarity Systems (LCS)

$$-y \in N_{\mathbb{R}^m_+}(\lambda) \Longleftrightarrow 0 \leqslant y \perp \lambda \geqslant 0 \tag{8}$$

- $K = [-1, 1]^m$, then we obtain linear relay systems (related to Filippov's DI and sliding mode control).

$$-y \in N_{[-1,1]^m}(\lambda) \Longleftrightarrow \lambda \in \mathrm{sgn}(y) \tag{9}$$

Objectives of my talk

- Understand what can be the nature of the solutions (uniqueness, smoothness).
- How perform the numerical time–integration ?
- Open issues for the time–integration of large dynamical systems arising in electrical network applications.

Introduction & Motivations

Mathematical nature of the solutions

Numerical time–integration

Open issues

Dynamical Complementarity systems

Definition (Linear Complementarity Systems (LCS))

A Linear Complementarity System (LCS) is defined by

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{10}$$

for the quadruplet $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$.

The notion of relative degree. Well-posedness

Definition (Relative degree in the SISO case ($m = 1$))

Let us consider a linear system in state representation given by the quadruplet $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{m\times n} \times \mathbb{R}$:

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \end{cases} \tag{11}$$

In the Single Input/ Single Output (SISO) case ($m = 1$), the relative degree r is defined by the first non zero Markov parameter :

$$D, CB, CAB, CA^2B, \ldots, CA^{r-1}B, \ldots \tag{12}$$

The notion of relative degree. Well-posedness

Definition (Uniform relative degree in the MIMO case ($m > 1$))

Let us consider a linear system in state representation given by the quadruplet $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$:

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \end{cases} \tag{11}$$

In the multiple input/multiple output (MIMO) case ($m > 1$), an *uniform* relative degree is defined as follows:

- If D is non singular, the relative degree is equal to 0.
- If $D = 0$, it is assumed to be the first positive integer r such that

  $$CA^iB = 0, \quad i = 0 \ldots r-2 \tag{12}$$

  while

  $$CA^{r-1}B \text{ is non singular.} \tag{13}$$

- Other cases yield a nonuniform relative degree.

The notion of relative degree. Well-posedness

Interpretation with the Markov parameters

The Markov parameters arise naturally when we derive with respect to time the output y,

$$\begin{array}{rcl} y & = & Cx + D\lambda \\ \dot{y} & = & CAx + CB\lambda,\ \text{if } D = 0 \\ \ddot{y} & = & CA^2x + CAB\lambda,\ \text{if } D = 0, CB = 0 \\ & \ldots & \\ y^{(r)} & = & CA^r x + CA^{r-1}B\lambda,\ \text{if } D = 0, CB = 0, CA^{r-2}B = 0, r = 1 \ldots r-2 \\ & \ldots & \end{array}$$

and the first non zero Markov parameter allows us to define the output y directly in terms of the input λ.

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The notion of relative degree. Well-posedness

Interpretation in terms of differential index of DAE

Let us consider the differential index ν of the following linear time–invariant DAE

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ y = 0 \end{cases} \tag{11}$$

If the uniform relative degree of the quadruplet (A, B, C, D) is r, then the differential index $\nu = r + 1$.

The notion of relative degree. Well-posedness

Example (Third relative degree LCS)

$$\begin{cases} \dddot{x}(t) = \lambda(t) - 1, \\ y(t) = x(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{12}$$

with the initial conditions, $x(0) = x_0 \geqslant 0, \dot{x}(t_0) = \dot{x}_0, \ddot{x}(t_0) = \ddot{x}_0$.

- the relative degree r is obviously 3, since $y^{(3)} = \dddot{x} = \lambda - 1$
- since $x_0 \geqslant 0$ satisfies the constraint, the function $x : [0, T] \rightarrow \mathbb{R}$ is usually assumed to be an absolutely continuous function of time.

The notion of relative degree. Well-posedness

Example (Third relative degree LCS)

Let us consider the dynamics at t_* when the constraint $y = x \geqslant 0$ becomes active, i.e., $x(t_*) = 0$,

- If $\dot{x}(t_*^-) > 0$, the system will instantaneously leaves the constraints with $\dot{x}(t_*^+) = \dot{x}(t_*^-) > 0$.
- If $\dot{x}(t_*^-) < 0$, the velocity $\dot{x}$ needs to jump to respect the constraint in $t + *^+$. (B.V. function)
- If $\dot{x}(t_*^-) < 0, \ddot{x}(t_*^-) < 0$, the velocity and the acceleration need to jump to respect the constraint in t^+. (Dirac + B.V. function)

➜ In the latter case, $\dddot{x}$ and λ must be considered as derivative of Dirac distribution.

Well–posedness for $r \geqslant 1$

What is the meaning of $\lambda \geqslant 0$ when $\lambda = \delta^{(r-1)}$

- If the initial conditions do not satisfy the constraints, the relative degree $r \geqslant 1$ needs a rigorous definition of the sign of a distribution. (see[Acary et al., 2008] for details)

➜ In this talk, we will focus on LCS of relative degree $r \leqslant 1$. The passive linear systems are of relative degree $\geqslant 1$ [Camlibel, 2001, Heemels and Brogliato, 2003]).

The notion of relative degree. Well-posedness

Example (The relative degree not sufficient [Heemels and Brogliato, 2003])

Let us consider the following LCS

$$\begin{cases} \dot{x} = -x + \lambda \\ y = x - \lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{12}$$

This system is strictly equivalent to

$$\dot{x} = \begin{cases} -x, & \text{if } x \geqslant 0 \\ 0, & \text{if } x \geqslant 0 \end{cases} \tag{13}$$

which leads to non existence of solutions for $x(0) < 0$ and to non uniqueness for $x(0) > 0$.

The notion of relative degree. Well-posedness

Possible further assumptions for existence and uniqueness

- The Rational Complementarity problem [Heemels, 1999, Camlibel, 2001, Camlibel et al., 2002]. The P-matrix property plays henceforth a fundamental role and provides the existence of global solution of the LCS in the sense of Caratheodory.
- For the relative degree $r = 1$ case, $\exists P > 0$, such that $PB = C^T$. A standard monotone differential inclusion is retrieved such that

$$-\left[\dot{z} + f(z,t)\right] \in A(z) \tag{12}$$

where A is a maximal monotone operator and f Lipschitz continuous.
➜ Existence and uniqueness of a solution $u \in C^0$ and $\dot{u} \in L^\infty$.
[Bastien and Schatzman, 2002]

To summarize

C^1 solutions

Mainly when $B\lambda$ singleton

- D positive definite (relative degree 0)
- D is a P-matrix
- D is a co-positive matrix

Absolutely continuous solutions

- relative degree 1 with $CB > 0$ or $PB = C^T$ and $P > 0$
- consistent initial conditions

Solution of Bounded Variations

- relative degree 1 with $CB > 0$ or $PB = C^T$ and $P > 0$
- inconsistent initial conditions

Open issues of the well–posedness

Cases that are taken into account by the uniform relative degree (MIMO case)

For instance, D or CB singular but non zero

- Generalize the differential index ?

Assumption of positive definiteness of the Markov parameters.

- General assumption equivalent to $B.SOL(Cx, D)$ for the relative degree 0.

Introduction & Motivations

Mathematical nature of the solutions

Numerical time–integration

Open issues

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Targeted system

$$
\begin{array}{ll}
\dot{x} = Ax + u(t) + r & \Big] \text{ Differential Equations} \\[2ex]
\begin{array}{l} y = Cx + D\lambda + a(t) \\ R = B\lambda \end{array} & \Big] \begin{array}{l} \text{Input/output relations} \\ \text{(nonsmooth components)} \end{array} \\[2ex]
0 \in y + N_{\mathbb{R}^m_+}(\lambda) &] \text{ Generalized equation} \\[2ex]
x(t_0) = x_0 &] \text{ Initial conditions}
\end{array}
\tag{13}
$$

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Time–stepping schemes. Design principles.

First principle

The fully implicit evaluation of the generalized equation in (13), that is on $[t_k, t_{k+1}]$,

$$0 \in y_{k+1} + N_{\mathbb{R}^m_+}(\lambda_{k+1}) \tag{14}$$

Second principle

A consistent evaluation of the unknown variables and their derivatives according to their smoothness.
For instance, time–stepping schemes must not approximate high order time–derivatives of functions which are not sufficiently smooth or must not try to point–wisely evaluate distributions.

Time–stepping schemes for a solution for class C^1

Required assumptions

- $B\lambda$ Lipschitz continuous function of x.
- In particular,

$$0 \in Cx + D\lambda + a + N_K(\lambda) \tag{15}$$

possesses a unique solution for all x.

Proposed scheme

$$\begin{cases} x_{k+1} - x_k = h\left(Ax_{k+\theta} + u_{k+\theta} + r_{k+\gamma}\right), \\ \\ y_{k+1} = Cx_{k+1} + D\lambda_{k+1} + a_{k+1}, \\ r_{k+1} = B\lambda_{k+1}, \\ \\ 0 \in y_{k+1} + N_K(\lambda_{k+1}), \end{cases} \tag{16}$$

with $\theta \in [0,1]$ and $\gamma \in [0,1]$. $x_{k+\theta} = (1-\theta)x_k + \theta x_{k+1}$
The initial value of $\lambda_0 = \lambda(t_0)$ is given by the solution of (15).

Time–stepping schemes for a solution for class C^1

One–step LCP

$$\begin{cases} y_{k+1} = M\lambda_{k+1} + q \\ 0 \leqslant y_{k+1} \perp \lambda_{k+1}) \geqslant 0, \end{cases} \tag{17}$$

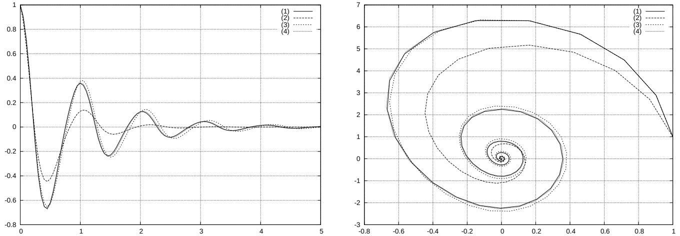
with

$$M = D + h\gamma C(I - h\theta A)^{-1}B, \tag{18}$$

and

$$q = a_{k+1} + C(I - h\theta A)^{-1}\left[(I + h(1-\theta)A)x_k + hu_{k+\theta} + h(1-\gamma)B\lambda_k\right]. \tag{19}$$

Time–stepping schemes for a solution for class C^1



(a) State x_1 vs. time. (b) Phase portrait. x_1 vs. x_2.

Figure: Solution of the RLC circuit with the time–stepping scheme (16). (1) Exact solution $x(t_k)$. (2) x_k with $\theta = 1, \gamma = 1$. (3) x_k with $\theta = 1/2, \gamma = 1$. (4) x_k with $\theta = 1/2, \gamma = 1/2$.

Influence of θ and γ

- Numerical parameters allows us to control numerical damping and order
- Second order accuracy can be achieved with $\theta = \gamma = 1/2$.
- Higher order approximations are useless and generate instabilities.

Time–stepping schemes for an absolutely continuous solution

Required assumptions

- Relative degree equal to 1 with consistent initial conditions

$$Cx_0 + a(t_0) \in \mathbb{R}^m_+. \tag{20}$$

  and smooth function $a(\cdot)$
- Monotony, positive definiteness, passivity assumption, ...

Proposed scheme

$$\begin{cases} x_{k+1} - x_k = h\left(Ax_{k+\theta} + u_{k+\theta} + r_{k+1}\right), \\ \\ y_{k+1} = Cx_{k+1} + D\lambda_{k+1} + a_{k+1}, \\ r_{k+1} = B\lambda_{k+1}, \\ \\ 0 \in y_{k+1} + N_K(\lambda_{k+1}), \end{cases} \tag{21}$$

with $\theta \in [0, 1]$.

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Time–stepping schemes for an absolutely continuous solution

Properties

- Order 1 is achieved for $\theta \in [0, 1]$
- θ controls the numerical damping and the stability

Why do not use $\gamma = 1/2$?

- No improvements for the order of accuracy
- Severe instabilities and numerical artifacts on r and λ.

With $\gamma = 1/2$, we attempt a second order approximation of a function of bounded variations λ.

Time–stepping schemes for solution of bounded variations

Required assumptions

- Relative degree equal to 1 with inconsistent initial conditions

$$Cx_0 + a(t_0) \notin \mathbb{R}^m_+. \tag{22}$$

  and/or nonsmooth function $a(\cdot)$.
- Monotony, positive definiteness, passivity assumption, ...
- Relative degree 2 with consistent initial conditions.

Time–stepping schemes for solution of bounded variations

Measure differential equations (in a nutshell)

$$dx = Ax(t)dt + u(t)dt + Bdi, \tag{23}$$

- dx is the differential measure associated with the RCBV function $\dot{x}(t)$ and di is also a measure
- The absolutely continuous function $\lambda(t)$ is the Radon-Nikodym derivative of di with respect to the Lebesgue measure, *i.e.* :

$$\frac{di}{dt} = \lambda(t). \tag{24}$$

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Time–stepping schemes for solution of bounded variations

Measure decomposition

$$di = \lambda(t)dt + \sum_i \sigma_i \delta_{t_i} \tag{25}$$

where δ_{t_i} is the Dirac measure at time of discontinuities t_i and σ_i the amplitude.

- Smooth dynamics :

$$\dot{x}(t) = Ax(t) + u(t) + B\lambda(t), \quad dt - \text{almost everywhere}, \tag{26}$$

- Jump dynamics at t_i :

$$x(t_i^+) - x(t_i^-) = B\sigma_i. \tag{27}$$

Time–stepping schemes for solution of bounded variations

Design of a consistent scheme

Only the measure of the time–intervals $(t_k, t_{k+1}]$ are considered such that :

$$dx((t_k, t_{k+1}]) = \int_{t_k}^{t_{k+1}} Ax(t) + u(t)\,dt + Bdi((t_k, t_{k+1}]). \tag{28}$$

By definition of the differential measure, we get

$$dx((t_k, t_{k+1}]) = x(t_{k+1}^+) - x(t_k^+). \tag{29}$$

The measure of the time–interval by *di* is kept as an unknown variable denoted by

$$\sigma_{k+1} = di((t_k, t_{k+1}]). \tag{30}$$

Finally, the remaining Lebesgue integral in (28) is approximated by an implicit Euler scheme

$$\int_{t_k}^{t_{k+1}} Ax(t) + u(t)\,dt \approx hAx_{k+1} + u_{k+1}. \tag{31}$$

Time–stepping schemes for solution of bounded variations

Proposed scheme

$$\begin{cases} x_{k+1} - x_k = h\,(Ax_{k+1} + u_{k+1}) + \sigma_{k+1}, \\ \\ y_{k+1} = Cx_{k+1} + a_{k+1}, \\ r_{k+1} = B\sigma_{k+1}, \\ \\ 0 \in y_{k+1} + N_K(\sigma_{k+1}). \end{cases} \tag{32}$$

Properties

- At best order 1 is achieved. (no rigorous proof for a finite accumulation of jumps)
- Implicitly implements a restitution rule with "no rebounds"

Time–stepping schemes for solution of bounded variations

Example (An RLD circuit)

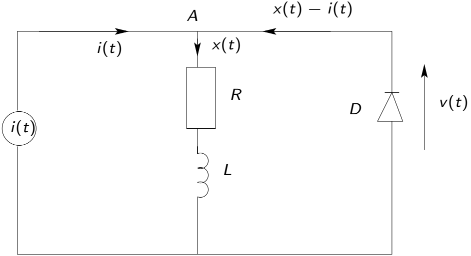


Figure: A circuit with an ideal diode, a resistor, an inductor and a current source.

$$\begin{cases} \dot{x}(t) = -\dfrac{R}{L}x(t) + v(t) \\ 0 \leqslant w(t) = x(t) - i(t) \perp v(t) \geqslant 0 \\ x(t^+) = i(t^+) + \max[0, x(t^-) - i(t^+)] \ \text{ at jump times,} \end{cases} \tag{33}$$

Time–stepping schemes for solution of bounded variations

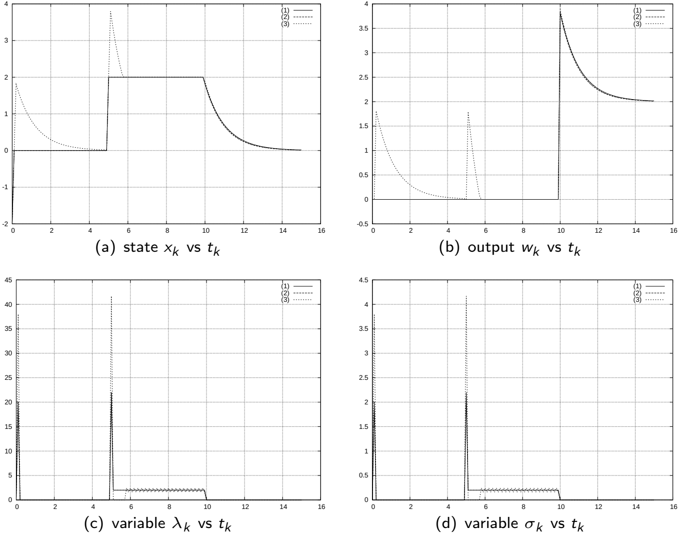
Let us choose $x(0^-) = -2$ and

$$i(t) = \begin{cases} 0 & \text{for all } t \in [0,5) \\ 2 & \text{for all } t \in (5,10) \\ -2 & \text{for all } t \geqslant 10 \end{cases}$$

The analytical solution with this value for the current source is:

$$x(t) = \begin{cases} x(0^+) = 0, & x(t) = 0, & v(t) = 0 & \text{on } t \in (0,5) \\ x(5^+) = 2, & x(t) = 2, & v(t) = 2 & \text{on } t \in [5,10) \\ x(10^+) = 2, & x(t) = 2e^{(10-t)} & v(t) = 0 & \text{on } t \in [10,+\infty). \end{cases} \tag{34}$$

Time–stepping schemes for solution of bounded variations



(a) state x_k vs t_k (b) output w_k vs t_k

(c) variable λ_k vs t_k (d) variable σ_k vs t_k

Figure: Simulation of system (33). (1) sheme (30). (2) scheme ((21)) with $\theta = 1/2$. (3) scheme (16) with $\theta = 1/2, \gamma = 1/2$.

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Open issues

Applications to industrial electrical networks

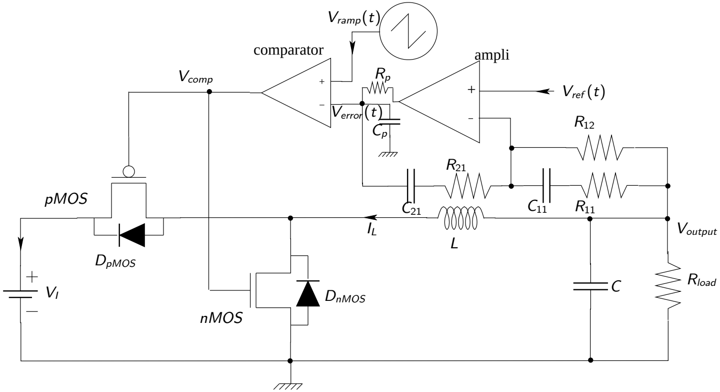
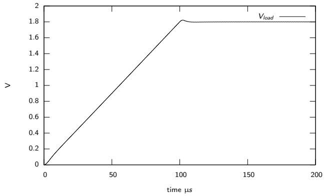
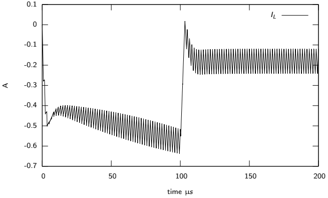


Figure: Buck converter.

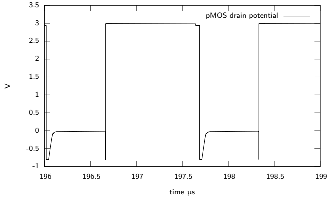
Applications to industrial electrical networks



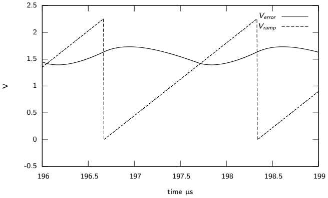
(a) V_{load}



(b) I_L



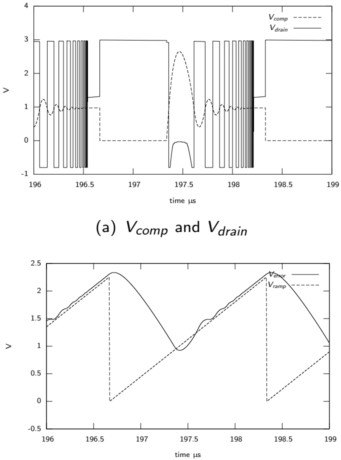
(c) pMOS drain potential



(d) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using standard parameters.

Applications to industrial electrical networks



(a) V_{comp} and V_{drain}

(b) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using sliding mode parameters.

Open issues

For more general formulations and more complex systems, are we able to infer the nature of the solutions? That is to say,

- Define and predict an equivalent notion to index and relative degree for instance, for a matrix D semi-definite positive.
- Given passive components, are we able to forecast the nature of the solutions from some topological considerations ? (as for the DAE case.)
- Adapt the time–stepping schemes in an hierarchical way in taking into account the "index" of each variable.

Introduction & Motivations

Mathematical nature of the solutions

Numerical time–integration

Open issues

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Thank you for your attention.

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